

TWO DIMENSIONAL INVISIBILITY CLOAKING VIA TRANSFORMATION OPTICS

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ABSTRACT. We investigate two-dimensional invisibility cloaking via transformation optics approach. The cloaking media possess much more singular parameters than those having been considered for three-dimensional cloaking in literature. Finite energy solutions for these cloaking devices are studied in appropriate weighted Sobolev spaces. We derive some crucial properties of the singularly weighted Sobolev spaces. The invisibility cloaking is then justified by decoupling the underlying singular PDEs into one problem in the cloaked region and the other one in the cloaking layer. We derive some completely novel characterizations of the finite energy solutions corresponding to the singular cloaking problems. Particularly, some ‘hidden’ boundary conditions on the cloaking interface are shown for the first time. We present our study for a very general system of PDEs, where the Helmholtz equation underlying acoustic cloaking is included as a special case.

1. INTRODUCTION

A region is said to be *cloaked* if its contents together with the cloak are indistinguishable from background space to certain exterior detections. Blueprints for making objects invisible to electromagnetic waves were proposed by Pendry *et al.* [17] and Leonhardt [11] in 2006. In the case of electrostatics, the same idea was discussed by Greenleaf *et al.* [5] in 2003. The works [5, 11, 17] rely on *transformation optics* for the construction of cloaking devices, which we shall further examine in the present paper. For state-of-the-art surveys on the rapidly growing literature and many applications of transformation optics, we refer to [4, 16, 19].

In transformation optics, the key ingredient is that optical material parameters have transformation properties which could be *pushed forward* to form new material parameters. Then, to construct cloaking devices, the idea is to *blow up* a point in the background space to form the cloaked region. The ambient background medium is then *pushed forward* to form the cloaking medium. Since the blowing-up transformation is singular, the resulting cloaking medium is inevitably singular. Two theoretical approaches have

Date: May 05, 2010.

Key words and phrases. invisibility cloaking, transformation optics, finite energy solutions, singularly weighted Sobolev space.

The work of HYL is partly supported by NSF grants, FRG DMS 0554571 and DMS 0758537.

come out to handle the singular cloaking problems. In Kohn *et al.* [10], the authors introduce the notion of *near-invisibility* cloaking in electrostatics from a regularization viewpoint. The singular ideal cloaking is regarded as the limit of the regular near-invisibility cloaking depending on certain regularizer. For acoustic cloaking, the near-invisibility is investigated in [9, 12, 15] in both two and three dimensions. In [3], Greenleaf *et al.* proposed to investigate the physically meaningful solutions, i.e. *finite energy solutions*, corresponding to degenerate differential equations underlying the three-dimensional cloaking. The proposal has been shown to work for both acoustic and electromagnetic cloaking, and can treat cloaking of passive objects as well as active/radiating objects. On the other hand, the analysis in [3] is conducted in the geometric setting by taking advantage of the one-to-one correspondence in \mathbb{R}^3 between the optical parameters and a smooth Riemannian metric. This argument does not carry over to \mathbb{R}^2 . As one shall see in Section 2, for two-dimensional cloaking, the cloaking medium has both degeneracy and blow-up singularities at the cloaking interface, making the problem more difficult to analyze.

In this paper, we consider the two-dimensional invisibility cloaking for a very general system of second order partial differential equations. The Helmholtz equation underlying the acoustic cloaking is included as a special case. In order to handle the singular cloaking problem, we follow the finite energy solutions approach from [3]. Recently, Hetmaniuk and Liu [6] introduce weighted Sobolev spaces with degenerate weights for three-dimensional acoustic cloaking problems, encompassing and generalizing the idea of finite energy solutions approach. For the present two-dimensional cloaking problems, we study weighted Sobolev spaces with more severely singular weights. The invisibility justification follows by study of weak solutions from the introduced weighted Sobolev space to the underlying singular PDEs. Our analysis is given in a very general setting. The cloaking is shown to work for arbitrary positive frequency and can cloak both passive media and source/sink inside the cloaked region. Since the cloaking media are more singular than those for the three-dimensional cloaking, we derive completely different and novel characterizations of solutions to the underlying wave equations compared to those derived for the three-dimensional acoustic cloaking in [3, 6]. Moreover, some ‘hidden’ boundary conditions on the cloaking interface are shown for the first time, giving more insights into the invisibility cloaking.

In this paper, we focus entirely on transformation-optics-approach in constructing cloaking devices. But we mention in passing the other promising cloaking schemes including the one based on anomalous localized resonance [14], and another one based on special (object-dependent) coatings [1]. The rest of the paper is organized as follows. In Section 2, we introduce the transformation optics and invisibility cloaking and give the construction of the two-dimensional radial cloaking devices. Section 3 is devoted to the

analysis of the radial cloaking by considering finite energy solutions in singularly weighted Sobolev spaces. In Section 4, we extend our study to general invisibility cloaking.

2. TRANSFORMATION OPTICS AND RADIAL INVISIBILITY CLOAKING

We first fix notations for some function spaces which are crucial for our study. Let Ω be a bounded Lipschitz domain in \mathbb{R}^2 . Let $m \geq 1$ be an integer, and $u : \Omega \rightarrow \mathbb{C}^m$ be a complex vector-valued function. $L^p(\Omega)^m$ is the space consisting of \mathbb{C}^m -valued measurable functions whose components belong to $L^p(\Omega)$. Following Schwartz, put $\mathcal{E}(\Omega)^m = C^\infty(\Omega)^m$ and $\mathcal{D}(\Omega)^m = C_{comp}^\infty(\Omega)^m$ be complex-valued, smooth test function spaces. Let $H^s(\Omega)$ denote the standard Sobolev space of order s and

$$\tilde{H}^s(\Omega) = \text{closure of } \mathcal{D}(\Omega) \text{ in } H^s(\mathbb{R}^n).$$

Note that $\tilde{H}^{-s}(\Omega)$ is an isometric realization of $H^s(\Omega)^*$ for $s \in \mathbb{R}$. The definition of the vector Sobolev spaces on Ω , $H^s(\Omega)^m = H^s(\Omega; \mathbb{C}^m)$ and $\tilde{H}^s(\Omega)^m = \tilde{H}^s(\Omega; \mathbb{C}^m)$ etc. shall now be obvious.

Next, we introduce the second-order partial differential operator (PDO) \mathcal{P} of the form

$$(2.1) \quad \mathcal{P}u = - \sum_{\alpha, \beta=1}^2 \frac{\partial}{\partial x_\alpha} \left(A^{\alpha\beta} \frac{\partial u}{\partial x_\beta} \right) - \omega^2 B u \quad \text{on } \Omega,$$

where the coefficients

$$(2.2) \quad A^{\alpha\beta} = [a_{pq}^{\alpha\beta}] \in L^\infty(\Omega)^{m \times m}, \quad B = [b_{pq}] \in L^\infty(\Omega)^{m \times m}$$

are functions from Ω into $\mathbb{C}^{m \times m}$, the space of complex $m \times m$ matrices. Here $\omega \in \mathbb{R}$ and $m \geq 1$ is an integer and thus, \mathcal{P} acts on a (column) vector-valued function $u : \Omega \rightarrow \mathbb{C}^m$ to give a vector-valued function $\mathcal{P}u : \Omega \rightarrow \mathbb{C}^m$, whose components are

$$(\mathcal{P}u)_p = - \sum_{k=1}^m \sum_{\alpha, \beta=1}^2 \partial_\alpha \left(a_{pk}^{\alpha\beta} \partial_\beta u_k \right) - \omega^2 \sum_{q=1}^m b_{pq} u_q.$$

In the sequel, let $A := [A^{\alpha\beta}] \in \mathbb{C}^{2m \times 2m}$ be the block matrix, and $\mathcal{P}_{[A, B]}$ be the PDO (2.1) associated with A and B . For the present study, we always assume that

$$(2.3) \quad (A^{\alpha\beta})^* = A^{\beta\alpha} \quad \text{for } \alpha, \beta = 1, 2,$$

where $*$ denotes the conjugate transpose of a matrix or vector. Moreover, we introduce the following algebraic conditions for the coefficient matrices $A^{\alpha\beta}$ and B . Let $0 < c_1 < c_2 < \infty$ be two constants. For all $x \in \Omega$, and

arbitrary $\xi_1, \xi_2 \in \mathbb{C}^m$ and $\eta \in \mathbb{C}^m$, we have

$$(2.4) \quad c_1 \sum_{l=1}^2 |\xi_l|^2 \leq \Re \sum_{\alpha, \beta=1}^2 [A^{\alpha\beta}(x) \xi_\beta]^* \xi_\alpha \leq c_2 \sum_{l=1}^2 |\xi_l|^2,$$

$$(2.5) \quad c_1 |\eta|^2 \leq \Re[B\eta]^* \eta \leq c_2 |\eta|^2.$$

Next, we associate $\mathcal{P}_{[A,B]}$ with a sesquilinear form $\mathcal{Q}_{[A,B]}$, defined by

$$(2.6) \quad \mathcal{Q}_{[A,B]}(u, v) = \int_{\Omega} \left(\sum_{\alpha, \beta=1}^2 (A^{\alpha\beta} \partial_\beta u)^* \partial_\alpha v - (Bu)^* v \right) dx.$$

Now, we are ready to present the PDE system for our study,

$$(2.7) \quad \mathcal{P}_{[A,B]} u = f \quad \text{on } \Omega,$$

where $f \in \tilde{H}^{-1}(\Omega)^m$. The Dirichlet-to-Neumann (DtN) map

$$(2.8) \quad \Lambda_{A,B,f}^\omega : H^{1/2}(\partial\Omega)^m \rightarrow H^{-1/2}(\partial\Omega)^m,$$

associated with (2.7) is defined by

$$(2.9) \quad \Lambda_{A,B,f}^\omega(h) = \sum_{\alpha, \beta \leq 2} \nu_\alpha A^{\alpha\beta} \gamma(\partial_\beta u) \quad \text{on } \partial\Omega,$$

where $\nu = (\nu_\alpha)_{\alpha=1}^2$ is the outward unit normal of $\partial\Omega$, $u \in H^1(\Omega)^m$ solves (2.7) with $u|_{\partial\Omega} = h$ and γ is the trace operator for Ω . The weak solution in (2.9) is variationally given by

$$(2.10) \quad \mathcal{Q}_{[A,B]}(u, \varphi) = \langle f, \varphi \rangle_\Omega := \int_{\Omega} f^* \varphi \, dx.$$

Due to (2.4) and (2.5), we know that $\mathcal{Q}_{[A,B]}$ (and so is $\mathcal{P}_{[A,B]}$) is *coercive* on $H^1(\Omega)^m$ in the sense that

$$(2.11) \quad \Re \mathcal{Q}(u, u) \geq c_1 \|u\|_{H^1(\Omega)^m}^2 - c_2 \|u\|_{L^2(\Omega)^m}^2 \quad \text{for } u \in H^1(\Omega)^m.$$

Hence, (2.10) is uniquely solvable except at a discrete set of eigenvalues for ω^2 (see, e.g. [13]). In fact, we know that $\Lambda_{A,B,f}^\omega$ is a well-defined continuous and invertible operator provided (2.4) and (2.5) are satisfied and ω^2 avoids the (discrete set of) eigenvalues (cf. [13]).

In the case $m = 1$ and A, B are both real, (2.7) is the scalar Helmholtz equation. It describes the time-harmonic solutions $p(x) = u(x)e^{-i\omega t}$ of the scalar wave equation $p_{tt} - B^{-1} \nabla \cdot (A \nabla p) = f(x)e^{-i\omega t}$. Here f represents a source/sink inside the region Ω . A and B are the acoustic material parameters of the medium supported in Ω , related respectively to, density tensor and modulus. For a *regular* acoustic medium, (2.4) and (2.5) are the physical condition on the material parameters. According to our earlier discussion, these are also mathematical conditions to guarantee the well-posedness of the underlying Helmholtz equation. In the sequel, we let $\{\Omega; A, B, f\}$ denote the medium and the source/sink supported in Ω . This is the prototype problem of our present study. Moreover, we are concerned with the inverse

problems of identifying the inside object $\{\Omega; A, B, f\}$ by the exterior wave measurements, which are encoded into the DtN operator (2.8). The inverse problems have widespread practical applications in science and engineering, and have received extensive and intensive investigations in last years (see, e.g., [8, 18]). In this context, an invisibility cloaking device is introduced as follows (see also [3, 9]).

Definition 2.1. Let Ω and D be bounded domains in \mathbb{R}^2 with $D \Subset \Omega$. $\Omega \setminus \bar{D}$ and D represent, respectively, the cloaking region and the cloaked region. $\{\Omega \setminus \bar{D}; A_c, B_c, f_c\}$ is said to be an *invisibility cloaking device* for the region D with respect to the regular reference/background space $\{\Omega; A_b, B_b, f_b\}$ if

$$\Lambda_{A_e, B_e, f_e}^\omega = \Lambda_{A_b, B_b, f_b}^\omega \quad \text{for all } \omega > 0,$$

where the extended medium $\{\Omega; A_e, B_e\}$ and the extended source f_e are given by

$$\{\Omega; A_e, B_e, f_e\} = \begin{cases} \{\Omega \setminus \bar{D}; A_c, B_c, f_c\} & \text{in } \Omega \setminus \bar{D}, \\ \{D; A_a, B_a, f_a\} & \text{in } D, \end{cases}$$

with $\{D; A_a, B_a, f_a\}$ arbitrary but regular.

In Definition 2.1, $\{\Omega; A, B\}$ is regular means that A and B satisfy the algebraic conditions (2.4) and (2.5). According to Definition 2.1, we note that the cloaking device $\{\Omega \setminus \bar{D}; A_c, B_c, f_c\}$ makes the interior target object $\{D; A_a, B_a, f_a\}$ indistinguishable from the reference/background space $\{\Omega; A_b, B_b, f_b\}$ by exterior detections. In fact, the exterior observer would not even be aware that something is being hidden.

Next, we present the transformation properties of material parameters, which are the cruxes of the construction of cloaking devices via transformation optics approach. In the following, we let $y = F(x)$ be a diffeomorphism such that $F : \Omega \rightarrow \tilde{\Omega}$. Then, the push-forwards of material parameters are given by

$$F_*\{\Omega; A, B\} = \{\tilde{\Omega}; F_*A, F_*B\} := \{\tilde{\Omega}; \tilde{A}, \tilde{B}\},$$

with

$$(2.12) \quad \tilde{A}^{pq}(y) = \sum_{\alpha, \beta=1}^2 \frac{\partial y_p}{\partial x_\beta} \frac{\partial y_q}{\partial x_\alpha} A^{\alpha\beta}(x) J^{-1} \Big|_{x=F^{-1}(y)}, \quad p, q = 1, 2,$$

$$(2.13) \quad \tilde{B}(y) = B(x) J^{-1} \Big|_{x=F^{-1}(y)},$$

where $J := \det([\partial y_\alpha / \partial x_\beta])$, the determinant of the Jacobian of F . We shall make use of the following result

Lemma 2.2. Let $F : \Omega \rightarrow \tilde{\Omega}$ be a diffeomorphism. For $u, v \in H^1(\Omega)^m$, let $\tilde{u} = (F^{-1})^*u := u \circ F \in H^1(\tilde{\Omega})^m$ and $\tilde{v} = (F^{-1})^*v \in H^1(\tilde{\Omega})^m$. Then we have

$$(2.14) \quad \mathcal{Q}_{[A, B]}(u, v) = \mathcal{Q}_{[F_*A, F_*B]}(\tilde{u}, \tilde{v}).$$

Proof. It is verified directly by change of variables in integrations as follows

$$\begin{aligned}
\mathcal{Q}_{[A,B]}(u,v) &= \int_{\Omega} \left(\sum_{\alpha,\beta=1}^2 [A^{\alpha\beta} \frac{\partial u}{\partial x_{\beta}}]^* \frac{\partial v}{\partial x_{\alpha}} - (Bu)^* v \right) dx \\
&= \int_{\tilde{\Omega}} \sum_{p,q=1}^2 \left[\left(\sum_{\alpha,\beta=1}^2 \frac{\partial y_p}{\partial x_{\beta}} \frac{\partial y_q}{\partial x_{\alpha}} A^{\alpha\beta} / J \right) \frac{\partial \tilde{u}}{\partial y_p} \right]^* \frac{\partial \tilde{v}}{\partial y_q} - \left(\frac{B}{J} \tilde{u} \right)^* \tilde{v} dy \\
&= \mathcal{Q}_{[\tilde{A},\tilde{B}]}(\tilde{u},\tilde{v}).
\end{aligned}$$

The proof is completed. \square

In the rest of this section, based on the above transformation properties, we give the construction of the two-dimensional cloaking devices, which we shall investigate in subsequent sections. We start our study by considering the cloaking of the unit central disc. In Section 4, we shall indicate how to extend our study to the general case. In the following, we denote by \mathbf{B}_{ρ} the central disc of radius ρ and $\mathbf{S}_{\rho} := \partial \mathbf{B}_{\rho}$. Let $\{\mathbf{B}_2; A_b, B_b, f_b\}$ be the (regular) background/reference space, where f_b is supported away from $\{0\}$. Here, we always assume that there is no eigenvalue problem for (2.10) in the reference space, and hence there is a well-defined DtN operator on \mathbf{S}_2 , namely $\Lambda_{A_b, B_b, f_b}^{\omega}$. Consider the transformation F , defined by

$$(2.15) \quad F : \begin{cases} \mathbf{B}_2 \setminus \{0\} & \rightarrow \mathbf{B}_2 \setminus \bar{\mathbf{B}}_1 \\ x & \mapsto (1 + \frac{|x|}{2}) \frac{x}{|x|} \end{cases}$$

F blows up the origin in the reference space to \mathbf{B}_1 while maps $\mathbf{B}_2 \setminus \{0\}$ to $\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1$ and keeps \mathbf{S}_2 fixed. We note that the blow-up transformation (2.15) has been extensively investigated for the design of three-dimensional cloaking devices in the literature (see [4]). Under the transformation F , the ambient reference medium in $\mathbf{B}_2 \setminus \{0\}$ is then push-forwarded to form transformation medium in $\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1$ as follows

$$(2.16) \quad A_c(y) = F_* A_b(x) \quad \text{and} \quad B_c(y) = F_* B_b(x), \quad y \in \mathbf{B}_2 \setminus \bar{\mathbf{B}}_1.$$

In Section 3, we shall show that

Theorem 2.3. $\{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1; A_c, B_c, f_c\}$ with $f_c = J^{-1}(F^{-1})^* f_b$ and A_c and B_c given by (2.16) is an invisibility cloaking device for the region \mathbf{B}_1 with respect to the reference space $\{\mathbf{B}_2; A_b, B_b, f_b\}$. That is, for any extended object

$$\{\mathbf{B}_2; A_e, B_e, f_e\} = \begin{cases} \{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1; A_c, B_c, f_c\} & \text{in } \mathbf{B}_2 \setminus \bar{\mathbf{B}}_1, \\ \{\mathbf{B}_1; A_a, B_a, f_a\} & \text{in } \mathbf{B}_1, \end{cases}$$

where $\{\mathbf{B}_1; A_a, B_a\}$ is an arbitrary regular medium and $f_a \in \tilde{H}^{-1}(\mathbf{B}_1)^m$, we have

$$(2.17) \quad \Lambda_{A_e, B_e, f_e}^{\omega} = \Lambda_{A_b, B_b, f_b}^{\omega}.$$

In the sequel, we conveniently define the push-forward of the source term as

$$F_*f = \det(DF)(F^{-1})^*f.$$

So, the cloaking device in Theorem 2.3 is given by

$$\{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1; A_c, B_c, f_c\} = F_*\{\mathbf{B}_2 \setminus \{0\}; A_b, B_b, f_b\}.$$

Next, we derive the explicit expressions of material parameters of the cloaking medium in Theorem 2.3. By (2.12)-(2.13) and straightforward calculations, we have for $y \in \mathbf{B}_2 \setminus \bar{\mathbf{B}}_1$

$$(2.18) \quad \begin{aligned} A_c(y) = & \frac{|y| - 1}{|y|} \Pi(y) \otimes I_m A_b(F^{-1}(y)) \Pi(y) \otimes I_m \\ & + \Pi(y) \otimes A_b(F^{-1}(y)) (I - \Pi(y)) \otimes I_m \\ & + \frac{|y|}{|y| - 1} (I - \Pi(y)) A_b(F^{-1}(y)) (I - \Pi(y)) \\ & + (I - \Pi(y)) \otimes I_m A_b(F^{-1}(y)) \Pi(y) \otimes I_m, \end{aligned}$$

$$(2.19) \quad B_c(y) = \frac{4(|y| - 1)}{|y|} B_b(F^{-1}(y)),$$

where I_m is the $m \times m$ identity matrix and $\Pi(y) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the projection to the radial direction, defined by

$$\Pi(y)\xi = \left(\xi \cdot \frac{y}{|y|} \right) \frac{y}{|y|},$$

i.e. $\Pi(y)$ is represented by the symmetric matrix $|y|^{-2}yy^T$. For $y \in \mathbf{B}_2 \setminus \bar{\mathbf{B}}_1$, we let $\hat{y} := y/|y| = [\hat{y}_1, \hat{y}_2]^T \in \mathbf{S}_1$ and $\hat{y}^\perp = [\hat{y}_1^\perp, \hat{y}_2^\perp]^T \in \mathbf{S}_1$ be such that $\hat{y} \cdot \hat{y}^\perp = 0$. Let $\mathbf{1}_m \in \mathbb{R}^m$ with each entry being 1, and set $\xi_l = \hat{y}_l \mathbf{1}_m \in \mathbb{R}^m$ and $\xi_l^\perp = \hat{y}_l^\perp \mathbf{1}_m \in \mathbb{R}^m$ for $l = 1, 2$. Then, by straightforward calculations, together with the following facts

$$\Pi(y)\hat{y} = \hat{y} \quad (I - \Pi(y))\hat{y} = 0, \quad \Pi(y)\hat{y}^\perp = 0, \quad (I - \Pi(y))\hat{y}^\perp = \hat{y}^\perp,$$

we have

$$\begin{aligned} \sum_{p,q=1}^2 [A_c^{pq}(y)\xi_q]^* \xi_p &= \frac{|y| - 1}{|y|} \sum_{p,q=1}^2 [A_b^{pq}(F^{-1}(y))\xi_q]^* \xi_p, \\ \sum_{p,q=1}^2 [A_c^{pq}(y)\xi_q^\perp]^* \xi_p^\perp &= \frac{|y|}{|y| - 1} \sum_{p,q=1}^2 [A_b^{pq}(F^{-1}(y))\xi_q^\perp]^* \xi_p^\perp. \end{aligned}$$

Since A_b is a regular medium parameter, we have by (2.4) that as $|y| \rightarrow 1^+$

$$\sum_{p,q=1}^2 [A_c^{pq}(y)\xi_q]^* \xi_p \rightarrow 0 \quad \text{and} \quad \sum_{p,q=1}^2 [A_c^{pq}(y)\xi_q^\perp]^* \xi_p^\perp \rightarrow \infty.$$

Meanwhile, it is obvious that B_c does not satisfy (2.5). That is, the algebraic conditions (2.4) and (2.5) for a regular medium are violated by the cloaking medium $\{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1; A_c, B_c\}$, which exhibits both degeneracy and blow-up singularities as one approaches the cloaking interface \mathbf{S}_1^+ , i.e. from $\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1$. We remark that this is in sharp difference from the study in [3] for three-dimensional cloaking devices where one would encounter only degeneracy singularities.

3. FINITE ENERGY SOLUTIONS FOR SINGULAR PDES

Consider the differential equation underlying the cloaking problem in Theorem 2.3,

$$(3.1) \quad \mathcal{P}_{[A_e, B_e]} u = f_e \quad \text{on } \mathbf{B}_2, \quad u|_{\mathbf{S}_2} = h.$$

As we have seen in the end of Section 2, the cloaking medium $\{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1; A_c, B_c\}$ is singular. One has to be careful in defining the meaning of a solution to the singular PDEs system (3.1) (see also Remark 3.9 for the necessity of introducing a suitable class of weak solutions to (3.1) other than spatial H^1 -solutions). To that end, we define for $\phi(y) \in \mathcal{E}(\mathbf{B}_2)^m$,

$$\mathcal{E}_{[A, B]}(\phi) = \left| \int_{\Omega} \left(\sum_{\alpha, \beta=1}^2 (A^{\alpha\beta} \partial_{\beta} \phi)^* \partial_{\alpha} \phi + (B\phi)^* \phi \right) dy \right|^{1/2}.$$

We note that if $\{\mathbf{B}_2; A, B\}$ is a regular medium, then it is straightforward to verify

$$(3.2) \quad \mathcal{E}_{[A, B]}(\phi) \sim \|\phi\|_{H^1(\mathbf{B}_2)^m}.$$

Here and in the sequel, for two relations R_1, R_2 , $R_1 \sim R_2$ means that there exists two finite positive constants c_1, c_2 such that $c_1 R_1 \leq R_2 \leq c_2 R_1$. Also, in the following, for notational convenience, we shall frequently refer to $R_1 \lesssim R_2$ as $R_1 \leq c_2 R_2$. Next, one can verify directly that due to the blow-up singularity of A_e on \mathbf{S}_1^+ ,

$$(3.3) \quad \phi \in \mathcal{E}(\mathbf{B}_2)^m \quad \text{and} \quad \mathcal{E}_{[A_e, B_e]}(\phi) < \infty \quad \text{iff} \quad \frac{\partial \phi}{\partial \theta} \Big|_{\mathbf{S}_1} = 0.$$

Here and also in what follows, we make use of the standard polar coordinate $(y_1, y_2) \mapsto (r \cos \theta, r \sin \theta)$ in \mathbb{R}^2 . Hence, we set

$$(3.4) \quad \mathcal{T}^{\infty}(\mathbf{B}_2)^m := \{\phi \in \mathcal{E}(\mathbf{B}_2)^m; \frac{\partial \phi}{\partial \theta} \Big|_{\mathbf{S}_1} = 0\},$$

and

$$(3.5) \quad \mathcal{T}_0^{\infty}(\mathbf{B}_2)^m := \mathcal{T}^{\infty}(\mathbf{B}_2)^m \cap \mathcal{D}(\mathbf{B}_2)^m,$$

which are closed subspaces of $\mathcal{E}(\mathbf{B}_2)^m$. Set

$$W_e(y) := \frac{4(|y| - 1)}{|y|} I_m \quad \text{for } 1 < |y| < 2; \quad I_m \quad \text{for } |y| \leq 1.$$

We note by using the fact B_b is a regular material parameter, together with (2.19) and (2.5), that for $\phi \in \mathcal{T}^\infty(\mathbf{B}_2)$

$$(3.6) \quad \mathcal{E}_{[A_e, W_e]}(\phi) \sim \mathcal{E}_{[A_e, B_e]}(\phi).$$

On the other hand, it is easy to see that $\mathcal{E}_{[A_e, W_e]}(\cdot)$ defines a norm on $\mathcal{T}^\infty(\mathbf{B}_2)^m$. Let

$$(3.7) \quad H_{[A_e, B_e]}^1(\mathbf{B}_2)^m := \text{cl}\{\mathcal{T}^\infty(\mathbf{B}_2)^m; \mathcal{E}_{[A_e, W_e]}(\cdot)\},$$

that is, the closure of the linear function space $\mathcal{T}^\infty(\mathbf{B}_2)^m$ with respect to the singularly weighted Sobolev norm $\mathcal{E}_{[A_e, W_e]}(\cdot)$. Clearly, one can consider the elements in $H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$ as \mathbb{C}^m -valued measurable functions. Moreover, we have

Lemma 3.1. *The map*

$$\phi \mapsto D_{A_e}^\alpha \phi := \sum_{\beta=1,2} A_e^{\alpha\beta} \partial_\beta \phi, \quad \phi \in \mathcal{T}^\infty(\mathbf{B}_2)^m, \quad \alpha = 1, 2$$

has a bounded extension

$$(3.8) \quad D_{A_e}^\alpha : H_{[A_e, B_e]}^1(\mathbf{B}_2)^m \mapsto \mathcal{M}(\mathbf{B}_2; \mathbb{C}^m),$$

where $\mathcal{M}(\mathbf{B}_2; \mathbb{C}^m)$ represents the space of complex \mathbb{C}^m -valued Borel measures on \mathbf{B}_2 . Moreover, for $u \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$, we have in the sense of Borel measures

$$(3.9) \quad (D_{A_e}^\alpha u)(\mathbf{S}_1) = 0, \quad \alpha = 1, 2.$$

Proof. Let $\phi \in \mathcal{T}^\infty(\mathbf{B}_2)^m$, $g \in C(\mathbf{B}_2)^m$ and $\tilde{\phi} = F^* \phi$, $\tilde{g} = F^* g \in L^\infty(B_2)^m$. Then, it is straightforward to show that $D_{A_e}^\alpha \phi \in L^\infty(\mathbf{B}_2)^m$. Hence, it has that

$$\begin{aligned} \int_{\mathbf{B}_2} (D_{A_e}^\alpha \phi)^* g \, dy &= \int_{\mathbf{B}_2 \setminus \mathbf{S}_1} (D_{A_e}^\alpha \phi)^* g \, dy \\ &= \int_{\mathbf{B}_2 \setminus \{0\}} \left(\sum_{k,l \leq 1} \frac{\partial y_\alpha}{\partial x_k} \frac{\partial \tilde{\phi}}{\partial x_l} A_b^{kl} \right)^* \tilde{g} \, dx + \int_{\mathbf{B}_1} (D_{A_a}^\alpha \phi)^* g \, dx \end{aligned}$$

On \mathbf{B}_2 , one has $\partial y_\alpha / \partial x_k = \mathcal{O}(\frac{1}{r})$. This together with the facts that $\{\mathbf{B}_2; A_b, B_b\}$ and $\{\mathbf{B}_1; A_a, B_a\}$ are regular, we use (3.2) to further have

$$\begin{aligned} & \left| \int_{\mathbf{B}_2} (D_{A_e}^\alpha \phi)^* g \, dy \right| \\ & \lesssim \|\tilde{\phi}\|_{H^1(\mathbf{B}_2)^m} \|\tilde{g}/r\|_{L^2(\mathbf{B}_2)^m} + \|\phi\|_{H^1(\mathbf{B}_2)^m} \|g\|_{L^2(\mathbf{B}_1)^m} \\ (3.10) \quad & \lesssim [\mathcal{E}_{[A_b, B_b]}(\tilde{\phi}) + \mathcal{E}_{[A_a, B_a]}(\phi)] \|g\|_{C(\mathbf{B}_2)^m} \text{dist}(\text{supp}(g), \mathbf{S}_1) \end{aligned}$$

$$(3.11) \quad \lesssim \mathcal{E}_{[A_e, B_e]}(\phi) \|g\|_{C(\mathbf{B}_2)^m} \text{dist}(\text{supp}(g), \mathbf{S}_1)$$

$$\begin{aligned} (3.12) \quad & \lesssim \mathcal{E}_{[A_e, W_e]}(\phi) \|g\|_{C(\mathbf{B}_2)^m} \text{dist}(\text{supp}(g), \mathbf{S}_1) \\ & \lesssim \|\phi\|_{H_{[A_e, B_e]}^1(\mathbf{B}_2)^m} \|g\|_{C(\mathbf{B}_2)^m} \text{dist}(\text{supp}(g), \mathbf{S}_1). \end{aligned}$$

In the above inequalities, from (3.10) to (3.11), we have made use the following facts by using Lemma 2.2

$$(3.13) \quad \mathcal{E}_{[A_b, B_b]}(\tilde{\phi}) = \mathcal{Q}_{[A_b, -B_b]}(\tilde{\phi}, \tilde{\phi}) = \mathcal{Q}_{[F_* A_b, -F_* B_b]}(\phi, \phi) = \mathcal{E}_{[A_e, B_e]}(\phi),$$

whereas from (3.11) to (3.12), we have made use the equivalence (3.6). This proves the bounded extension (3.8). Finally, (3.9) follows by taking functions g supported in sufficiently small neighborhoods of \mathbf{S}_1 .

The proof is completed. \square

Now, the solution to the singular system of PDEs (3.1) is defined by the distributional duality as to find $u \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$ such that $u|_{\mathbf{S}_2} = h \in H^{1/2}(\mathbf{S}_2)^m$ and

$$(3.14) \quad \mathcal{Q}_{[A_e, B_e]}(u, \phi) = \langle f_e, \phi \rangle, \quad \forall \phi \in \mathcal{T}_0^\infty(\mathbf{B}_2)^m.$$

Remark 3.2. Since the singularities of A_e and B_e are only attached to \mathbf{S}_1^+ , we know that for any $u \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$, $u \in H_{loc}^1(\mathbf{B}_2 \setminus \mathbf{S}_1)^m$. Therefore, for (3.14) we have the well-defined $u|_{\mathbf{S}_2} = h \in H^{1/2}(\mathbf{S}_2)^m$ and also a well-defined Dirichlet-to-Neumann map on \mathbf{S}_2 defined by

$$(3.15) \quad \Lambda_{A_e, B_e, f_e}^\omega(h) = \sum_{\alpha, \beta \leq 1} \nu_\alpha A_e^{\alpha\beta} \partial_\beta u \in H^{-1/2}(\mathbf{S}_2)^m,$$

provided (3.14) has a unique solution.

Remark 3.3. As is known,

$$\left| \int_{\mathbf{B}_2} \left(\sum_{\alpha, \beta=1}^2 (A_e^{\alpha\beta} \partial_\beta u)^* \partial_\alpha u + (B_e u)^* u \right) dy \right|$$

is the (generalized) energy of the system. Hence, the solution in (3.14) is physically meaningful in that it has finite energy.

Next, we shall show the following on the solution of (3.14).

Theorem 3.4. $u \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$ is a solution to (3.14) if and only if $\tilde{v}(x) = (F^*(u|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1}))^e \in H^1(\mathbf{B}_2)^m$ with $(F^*(u|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1}))^e$ denoting the extension of $F^*(u|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1})$ from $\mathbf{B}_2 \setminus \{0\}$ to \mathbf{B}_2 (e.g., by setting it be 0), is a solution to

$$(3.16) \quad \mathcal{P}_{[A_b, B_b]} \tilde{v} = f_b \quad \text{on } \mathbf{B}_2, \quad \tilde{v}|_{\mathbf{S}_2} = h$$

and $w = u|_{\mathbf{B}_1} \in H^1(\mathbf{B}_1)^m$ is a solution to

$$(3.17) \quad \mathcal{P}_{[A_a, B_a]} w = f_a \quad \text{on } \mathbf{B}_1, \quad w|_{\mathbf{S}_1} = \mathbf{c}_0,$$

where $\mathbf{c}_0 \in \mathbb{C}^m$ is a constant vector determined by

$$(3.18) \quad \int_{\mathbf{S}_1} \sum_{\alpha, \beta \leq 1} \nu_\alpha A_a^{\alpha\beta} \partial_\beta w \, dS = 0.$$

As a direct consequence of Theorem 3.4, we first give the proof of Theorem 2.3.

Proof of Theorem 2.3. Let $u \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$ be a solution to (3.1) corresponding to the cloaking problem for $\{\mathbf{B}_2; A_e, B_e, f_e\}$, whereas $\tilde{v} \in H^1(\mathbf{B}_2)^m$ be a solution to (3.16) corresponding to the differential equation in the reference space $\{\mathbf{B}_2; A_b, B_b, f_b\}$. By Theorem 3.4, we have

$$u|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1} = (F^{-1})^*(\tilde{v}|_{\mathbf{B}_2 \setminus \{0\}}), \quad \tilde{v}|_{\mathbf{B}_2 \setminus \{0\}} = F^*(u|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1}).$$

Next, let $\kappa(t) \in \mathcal{D}(\mathbb{R})$ be a smooth real cut-off function such that $0 \leq \kappa(t) \leq 1$ with $\kappa(t) = 1$ for $t < 4/3$ and $\kappa(t) = 0$ for $t > 5/3$. For arbitrary $\phi \in \mathcal{E}(\mathbf{B}_2)$, set $\psi(y) := (1 - \kappa(|y|))\phi(y) \in \mathcal{T}^\infty(\mathbf{B}_2)$. By Green's identity, we have

$$\begin{aligned} \mathcal{Q}_{[A_c, B_c]}(u, \psi) &= \mathcal{Q}_{[A_e, B_e]}(u, \psi) \\ &= \langle f_e, \psi \rangle_{\mathbf{B}_2} + \langle \Lambda_{A_e, B_e, f_e}^\omega(h), \gamma\psi \rangle_{\mathbf{S}_2} \\ (3.19) \quad &= \langle f_c, \psi \rangle_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1} + \langle \Lambda_{A_e, B_e, f_e}^\omega(h), \gamma\psi \rangle_{\mathbf{S}_2}. \end{aligned}$$

Then, by change of variables in integrations in (3.19), and using Lemma 2.2 and, $F|_{\mathbf{S}_2} = \text{Identity}$ and $f_c = J^{-1}(F^{-1})^*f_b$, we further have

$$(3.20) \quad \mathcal{Q}_{[A_b, B_b]}(\tilde{v}, \tilde{\psi}) = \langle f_b, \tilde{\psi} \rangle_{\mathbf{B}_2} + \langle \Lambda_{A_e, B_e, f_e}^\omega(h), \gamma\psi \rangle_{\mathbf{S}_2},$$

where $\tilde{\psi} = F^*\psi \in \mathcal{E}(\mathbf{B}_2)$. By using Green's identity again, we know

$$\mathcal{Q}_{[A_b, B_b]}(\tilde{v}, \tilde{\psi}) = \langle f_b, \tilde{\psi} \rangle_{\mathbf{B}_2} + \langle \Lambda_{A_b, B_b, f_b}^\omega(h), \gamma\psi \rangle_{\mathbf{S}_2},$$

which implies by (3.20) that

$$\langle \Lambda_{A_e, B_e, f_e}^\omega(h), \gamma\psi \rangle_{\mathbf{S}_2} = \langle \Lambda_{A_b, B_b, f_b}^\omega(h), \gamma\psi \rangle_{\mathbf{S}_2}$$

and hence

$$\Lambda_{A_e, B_e, f_e}^\omega = \Lambda_{A_b, B_b, f_b}^\omega.$$

The proof is completed. \square

We proceed to the proof of Theorem 3.4. We first derive two auxiliary lemmata characterizing the singularly weighted Sobolev space $H_{[A_e, B_e]}^1(\mathbf{B}_2)$. In the following, χ_Ω denotes the characteristic function for a set $\Omega \subset \mathbb{R}^2$.

Lemma 3.5. *Let \mathbf{c}_1 and \mathbf{c}_2 be two constant vectors in \mathbb{C}^m . Then*

$$(3.21) \quad \mathbf{c}_1 \chi_{\bar{\mathbf{B}}_1} + \mathbf{c}_2 \chi_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1} \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m.$$

Proof. Since a constant function always belongs to $H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$, it suffices to show that $\varphi(y) := \chi_{\bar{\mathbf{B}}_1} \mathbf{c} \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$ with $\mathbf{c} \in \mathbb{C}^m$ a constant vector. Let $\rho \in \mathcal{D}(\mathbb{R})$ be a cut-off function such that $0 \leq \rho(t) \leq 1$ with $\rho(t) = 1$ for $t < 1/2$ and $\rho(t) = 0$ for $t > 1$. Then, define for $\varepsilon > 0$,

$$(3.22) \quad \varphi_\varepsilon(y) = \begin{cases} \mathbf{c}, & y \in \bar{\mathbf{B}}_1, \\ \rho(\frac{\ln \varepsilon}{\ln(|y|-1)}) \mathbf{c}, & y \in \mathbf{B}_2 \setminus \bar{\mathbf{B}}_1. \end{cases}$$

Obviously, $\varphi_\varepsilon \in \mathcal{T}^\infty(\mathbf{B}_2)^m$. Next, we shall show

$$(3.23) \quad \lim_{\varepsilon \rightarrow 0^+} \|\varphi_\varepsilon - \varphi\|_{H^1_{[A_e, B_e]}(\mathbf{B}_2)^m} = 0,$$

which then implies that $\varphi \in H^1_{[A_e, B_e]}(\mathbf{B}_2)^m$. In fact, we have

$$(3.24) \quad \begin{aligned} & \|\varphi_\varepsilon - \varphi\|_{H^1_{[A_e, B_e]}(\mathbf{B}_2)^m}^2 = \mathcal{E}_{[A_e, W_e]}^2(\varphi_\varepsilon - \varphi) \\ & \lesssim \mathcal{E}_{[A_e, B_e]}^2(\varphi_\varepsilon - \varphi) = \mathcal{E}_{[A_c, B_c]}^2(\varphi_\varepsilon) \\ & = \left| \int_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1} \left(\sum_{\alpha, \beta \leq 1} (A_c^{\alpha\beta} \partial_\beta \varphi_\varepsilon)^* \partial_\alpha \varphi_\varepsilon + (B_c \varphi_\varepsilon)^* \varphi_\varepsilon \right) dy \right| \end{aligned}$$

By using the explicit expression for B_c in (2.19) together with the fact the B_b is a regular material parameter, it is straightforwardly shown that

$$(3.25) \quad \begin{aligned} \left| \int_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1} (B_c \varphi)^* \varphi dy \right| & \lesssim \int_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1} \left| \rho\left(\frac{\ln \varepsilon}{\ln(|y| - 1)}\right) \right|^2 dy \\ & \lesssim \text{Vol}(\Gamma_\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+, \end{aligned}$$

where

$$\Gamma_\varepsilon := \{y \in \mathbb{R}^2; 1 < |y| < 1 + \varepsilon\}.$$

Next, by direct calculations, we have

$$(3.26) \quad \nabla \left(\rho\left(\frac{\ln \varepsilon}{\ln(|y| - 1)}\right) \right) = \rho'\left(\frac{\ln \varepsilon}{\ln(|y| - 1)}\right) \cdot \frac{\ln \varepsilon}{|\ln(|y| - 1)|^2} \cdot \frac{\hat{y}}{|y| - 1},$$

where $\hat{y} := y/|y| \in \mathbf{S}_1$. By using (2.18) and (3.26), we further have

$$(3.27) \quad \begin{aligned} & \left(\nabla \rho\left(\frac{\ln \varepsilon}{\ln(|y| - 1)}\right) \otimes \mathbf{c} \right)^* A_c^* \left(\nabla \rho\left(\frac{\ln \varepsilon}{\ln(|y| - 1)}\right) \otimes \mathbf{c} \right) \\ & = \left[\rho'\left(\frac{\ln \varepsilon}{\ln(|y| - 1)}\right) \right]^2 \cdot \frac{|\ln \varepsilon|^2}{|\ln(|y| - 1)|^4} \cdot \frac{1}{|y| - 1} \cdot [(A_b(\hat{y} \otimes \mathbf{c}))^* (\hat{y} \otimes \mathbf{c})] \end{aligned}$$

By using (3.27), we can deduce

$$(3.28) \quad \begin{aligned} & \left| \int_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1} \sum_{\alpha, \beta \leq 1} (A_c^{\alpha\beta} \partial_\beta \varphi_\varepsilon)^* \partial_\alpha \varphi_\varepsilon dy \right| \\ & = \left| \int_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1} \left(\nabla \rho\left(\frac{\ln \varepsilon}{\ln(|y| - 1)}\right) \otimes \mathbf{c} \right)^* A_c^* \left(\nabla \rho\left(\frac{\ln \varepsilon}{\ln(|y| - 1)}\right) \otimes \mathbf{c} \right) dy \right| \\ & \lesssim \left| \int_{\Gamma_\varepsilon} \frac{|\ln \varepsilon|^2}{|\ln(|y| - 1)|^4} \cdot \frac{1}{|y| - 1} dy \right| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+. \end{aligned}$$

Combining (3.24), (3.25) and (3.28), we have (3.23). The proof is completed. \square

Lemma 3.6. *Let u be a measurable \mathbb{C}^m -valued function on \mathbf{B}_2 . Then $u \in H^1_{[A_e, B_e]}(\mathbf{B}_2)^m$ if and only if the following two conditions hold:*

- (i) $w := u|_{\mathbf{B}_1} \in H^1(\mathbf{B}_1)^m$ and
- (3.29) $\gamma^- w := w|_{\mathbf{S}_1^-} = \text{constant},$
- where γ^- is the trace operator on \mathbf{S}_1^- , i.e. as one approaches \mathbf{S}_1 from \mathbf{B}_1 .
- (ii) $v := u|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1}$ satisfies

$$(3.30) \quad \left| \int_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1} \sum_{\alpha, \beta \leq 1} (A_c^{\alpha\beta} \partial_\beta v)^* \partial_\alpha v \, dy \right| < \infty, \quad \left| \int_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1} (B_c v)^* v \, dy \right| < \infty$$

and

$$(3.31) \quad e_\theta \cdot \nabla v_p|_{\mathbf{S}_1^+} = 0, \quad p = 1, 2, \dots, m,$$

where e_θ is the unit angular directional vector on the sphere \mathbf{S}_1 .

Proof. First, we show conditions (i) and (ii) are necessary for $u \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$.

Clearly, we have

$$\mathcal{E}_{[A_a, B_a]}(w) \lesssim \|u\|_{H_{[A_e, B_e]}^1(\mathbf{B}_2)^m},$$

which together with the fact $\{\mathbf{B}_1; A_a, B_a\}$ is regular implies that

$$\|w\|_{H^1(\mathbf{B}_1)^m} \lesssim \|u\|_{H_{[A_e, B_e]}^1(\mathbf{B}_2)^m}, \quad \text{i.e.,} \quad w \in H^1(\mathbf{B}_1)^m.$$

Let $\{\phi_n\}_{n=1}^\infty \subset \mathcal{T}^\infty(\mathbf{B}_2)^m$ be such that

$$\|\phi_n - u\|_{H_{[A_e, B_e]}^1(\mathbf{B}_2)^m} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then, we obviously have

$$\|\phi_n|_{\mathbf{B}_1} - w\|_{H^1(\mathbf{B}_1)^m} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence,

$$\|\gamma^- \phi_n - \gamma^- w\|_{H^{1/2}(\mathbf{S}_1)^m} \lesssim \|\phi_n|_{\mathbf{B}_1} - w\|_{H^1(\mathbf{B}_1)^m} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Noting $\partial \phi_n / \partial \theta|_{\mathbf{S}_1} = 0$, we know $\gamma^- \phi_n$ are constants independent of the angular variable θ . Therefore, $\gamma^- w$ is constant. Whereas for $v := u|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1}$, it is trivial to see that (3.30) holds. To see (3.31), we let $\tilde{v} = F^* v$. Using the change of variables in (3.30), we have

$$\left| \int_{\mathbf{B}_2 \setminus \{0\}} \sum_{\alpha, \beta \leq 1} (A_b^{\alpha\beta} \partial_\beta \tilde{v})^* \partial_\alpha \tilde{v} \, dx \right| < \infty \quad \text{and} \quad \left| \int_{\mathbf{B}_2 \setminus \{0\}} (B_b \tilde{v})^* \tilde{v} \, dx \right| < \infty.$$

Since the reference space $\{\mathbf{B}_2; A_b, B_b\}$ is regular, we further have by using (2.4) and (2.5)

$$\int_{\mathbf{B}_2 \setminus \{0\}} \sum_{\alpha \leq 1} |\partial_\alpha \tilde{v}|^2 \, dx < \infty \quad \text{and} \quad \int_{\mathbf{B}_2 \setminus \{0\}} |\tilde{v}|^2 \, dx < \infty.$$

Set $\Psi_p = \nabla \tilde{v}_p|_{\mathbf{B}_2 \setminus \{0\}}$, $p = 1, 2, \dots, m$. We have $\Psi_p \in L^2(\mathbf{B}_2 \setminus \{0\})^{2 \times 1}$. Extending Ψ and \tilde{v} on $\{0\}$ (e.g., by setting to be zero), and using the same notations for the extensions, we have

$$\Psi_p \in (L^2(\mathbf{B}_2))^{2 \times 1} \text{ and } \tilde{v}_p \in L^2(\mathbf{B}_2).$$

The difference $\nabla \tilde{v}_p - \Psi_p$ belongs to $H^{-1}(\mathbf{B}_2)$ and it is supported on $\{0\}$. Since no non-zero distribution in $H^{-1}(\mathbf{B}_2)$ is supported on $\{0\}$, we see $\nabla \tilde{v}_p = \Psi_p$. Hence, $\tilde{v}_p \in H^1(\mathbf{B}_2)$ and therefore $\tilde{v} \in H^1(\mathbf{B})^m$.

Let the Fourier decomposition of \tilde{v}_p , $1 \leq p \leq m$, be given by

$$(3.32) \quad \tilde{v}_p(r, \theta) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \tilde{v}_p^{(n)}(r) e^{in\theta}.$$

In [2], it is proved

$$(3.33) \quad \tilde{v}_p^{(n)}(0) = 0, \quad n \neq 0.$$

Formally, we write $\tilde{v}_p(0, \theta) = \tilde{v}_p^{(0)}(0)$. Since $v_p \chi_{B_2 \setminus \overline{B_1}} = (F^{-1})^* \tilde{v}_p$, we have

$$\forall 1 < r < 2, \quad v_p(r, \theta) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} v_p^{(n)}(r) e^{in\theta} \quad \text{with} \quad v_p^{(n)}(r) = \tilde{v}_p^{(n)}(2(r-1)).$$

By (3.33), we have

$$v_p^{(n)}(1) = \tilde{v}_p^{(n)}(0) = 0 \quad \text{for } n \neq 0,$$

which implies

$$(3.34) \quad \left. \frac{1}{r} \frac{\partial v_p}{\partial \theta}(r, \theta) \right|_{r=1} = 0.$$

Next, we show that (i) and (ii) are also sufficient conditions for a measurable function u on \mathbf{B}_2 to belong to $H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$. We first assume that $u|_{\mathbf{B}_2 \setminus \mathbf{B}_1} = 0$. By (i), let \mathbf{c}_0 be a constant vector such that $u|_{\mathbf{S}_1^-} = \mathbf{c}_0$. According to Lemma 3.5, it suffices to show that $\hat{u} := u - \mathbf{c}_0 \chi_{\mathbf{B}_1} \in H_{[A_e, B_e]}^1(\mathbf{B}_1)^m$. Clearly,

$$\hat{w} := \hat{u} \chi_{\mathbf{B}_1} = w - \mathbf{c}_0 \chi_{\mathbf{B}_1} \in H_0^1(\mathbf{B}_1)^m.$$

Hence, there exists $\{\varphi_n\}_{n=1}^\infty \subset \mathcal{D}(\mathbf{B}_1)^m$ such that

$$\|\varphi_n - \hat{w}\|_{H^1(\mathbf{B}_1)^m} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let $\phi_n \in \mathcal{F}^\infty(\mathbf{B}_2)^m$ be such that $\phi_n|_{\mathbf{B}_1} = \varphi_n$ and $\phi_n|_{\mathbf{B}_2 \setminus \mathbf{B}_1} = 0$. Using the fact that $\{\mathbf{B}_1; A_a, B_a\}$ is regular and $\hat{u}|_{\mathbf{B}_2 \setminus \overline{\mathbf{B}_1}} = 0$, we have

$$\begin{aligned} \|\phi_n - \hat{u}\|_{H_{[A_e, B_e]}^1(\mathbf{B}_2)^m}^2 &= \mathcal{E}_{[A_a, B_a]}^2(\varphi_n - \hat{w}) \\ &\lesssim \|\varphi_n - \hat{w}\|_{H^1(\mathbf{B}_1)^m}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which implies that $\hat{u} \in H_{(A_e, B_e)}^1(\mathbf{B}_1)^m$.

Now, let u be a measurable function satisfying (i) and (ii). As is shown above, $u \chi_{\mathbf{B}_1} \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$, one only needs to show that $u - u \chi_{\mathbf{B}_1} \in$

$H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$; that is, u vanishes inside \mathbf{B}_1 . By (3.31), we know $u|_{\mathbf{S}_1^+}$ is constant independent of the angular variable θ . Using similar argument as earlier by subtraction of a Heaviside function from u together with Lemma 3.5, we can further assume that u vanishes on $\bar{\mathbf{B}}_1$. Moreover, without loss of generality, we can also assume that u vanishes near \mathbf{S}_2 . Let $\tilde{v} := F * (u|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1}) = F^* v$. According to our earlier argument, $v \in H^1(\mathbf{B}_2)^m$ (note here we identify v and its extension on $\{0\}$). Since v vanishes near \mathbf{S}_2 and $\{0\}$ is a $(2, 1)$ -polar set, there are $\tilde{\phi}_n \in \mathcal{D}(\mathbf{B}_2 \setminus \{0\})^m$ such that

$$\|\tilde{\phi}_n - \tilde{v}\|_{H^1(\mathbf{B}_2)^m} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let $\phi_n \in \mathcal{T}^\infty(\bar{\mathbf{B}}_2)^m$ be such that

$$\phi_n|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1} = (F^{-1})^* \tilde{\phi}_n \quad \text{and} \quad \phi_n|_{\bar{\mathbf{B}}_1} = 0.$$

Using $u|_{\bar{\mathbf{B}}_1} = 0$ and the fact that $\{\mathbf{B}_2; A_b, B_b\}$ is regular, we have

$$\begin{aligned} \|\phi_n - u\|_{H_{[A_e, B_e]}^1(\mathbf{B}_2)^m} &\lesssim \mathcal{E}_{[\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1; A_e, B_e]}(\phi_n - v) \\ &= \mathcal{E}_{[A_b, B_b]}(\tilde{\phi}_n - \tilde{v}) \\ &\lesssim \|\tilde{\phi}_n - \tilde{v}\|_{H^1(\mathbf{B}_2)^m} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which implies that $u \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$.

The proof is completed. \square

Now, we are ready to present the proof of Theorem 3.4.

Proof of Theorem 3.4. We first show that if $u \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$ is a solution to (3.1) and (3.14), then we must have the decoupled problems (3.16) and (3.17).

Let $v := u|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1}$ and $\tilde{v} = F^* v$. In the following, as in the proof of Lemma 3.6, we identify \tilde{v} and its H^1 -extension from $\mathbf{B}_2 \setminus \{0\}$ to \mathbf{B}_2 . Clearly, \tilde{v} satisfies (3.16). On the other hand, let $w := u|_{\bar{\mathbf{B}}_1}$, then $\gamma^- w$ is a constant vector independent of the angular variable θ by Lemma 3.6. So, it is trivial to see that w satisfies (3.17). Hence, we only need to show (3.18), whereas the determination of \mathbf{c}_0 from (3.18) will be discussed in the subsequent Theorem 3.7.

Set

$$\Sigma_\varepsilon^+ := \{y \in \mathbb{R}^2; 1 < |y| < 1 + \varepsilon/2\}, \quad \Sigma_\varepsilon^- := \{y \in \mathbb{R}^2; 1 - \varepsilon/2 < |y| < 1\},$$

and

$$\Upsilon_\varepsilon^+ := \{y \in \mathbb{R}^2; |y| = 1 + \varepsilon/2\}, \quad \Upsilon_\varepsilon^- := \{y \in \mathbb{R}^2; |y| = 1 - \varepsilon/2\}.$$

We first show for $\varphi \in \mathcal{T}_0^\infty(\mathbf{B}_2)^m$

$$(3.35) \quad \lim_{\varepsilon \rightarrow 0^+} \left| \int_{\Sigma_\varepsilon^+} \left(\sum_{\alpha, \beta \leq 1} (A_e^{\alpha\beta} \partial_\beta u)^* \partial_\alpha \varphi - \omega^2 (B_e u)^* \varphi - f_e^* \varphi \right) dy \right| = 0.$$

In fact, by using the expression of (A_e, B_e) in Σ_ε^+ and the change of variables in integrations, together with f_e being $J^{-1}(F^{-1})^*f_b$ in Σ_ε^+ and \tilde{v} being the H^1 -extension of F^*v from $\mathbf{B}_2 \setminus \{0\}$ to \mathbf{B}_2 , we have

$$\begin{aligned}
(3.36) \quad & \left| \int_{\Sigma_\varepsilon^+} \left(\sum_{\alpha, \beta \leq 1} (A_e^{\alpha\beta} \partial_\beta u)^* \partial_\alpha \varphi - \omega^2 (B_e u)^* \varphi - f_e^* \varphi \right) dy \right| \\
&= \left| \int_{\mathbf{B}_\varepsilon} \left(\sum_{\alpha, \beta \leq 1} (A_b^{\alpha\beta} \partial_\beta \tilde{v})^* \partial_\alpha \tilde{\varphi} - \omega^2 (B_b \tilde{v})^* \tilde{\varphi} - f_b^* \tilde{\varphi} \right) dx \right| \\
&\lesssim \left(\|\tilde{v}\|_{H^1(\mathbf{B}_\varepsilon)^m} + \|f_b\|_{\tilde{H}^{-1}(\mathbf{B}_\varepsilon)^m} \right) \|\tilde{\varphi}\|_{H^1(\mathbf{B}_\varepsilon)^m} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+,
\end{aligned}$$

where $\tilde{\varphi} \in H^1(\mathbf{B}_2)^m$ is the continuous extension of $F^*(\phi|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1})$ from $\mathbf{B}_2 \setminus \{0\}$ to \mathbf{B}_2 . On the other hand, noting $u|_{\mathbf{B}_1} = w \in H^1(\mathbf{B}_1)^m$ and $f_a|_{\mathbf{B}_1} \in \tilde{H}^{-1}(\mathbf{B}_1)^m$, it is straightforward to verify

$$\begin{aligned}
(3.37) \quad & \lim_{\varepsilon \rightarrow 0^+} \left| \int_{\Sigma_\varepsilon^-} \left(\sum_{\alpha, \beta \leq 1} (A_e^{\alpha\beta} \partial_\beta u)^* \partial_\alpha \varphi - \omega^2 (B_e u)^* \varphi - f_e^* \varphi \right) dy \right| \\
&= \lim_{\varepsilon \rightarrow 0^+} \left| \int_{\Sigma_\varepsilon^-} \left(\sum_{\alpha, \beta \leq 1} (A_a^{\alpha\beta} \partial_\beta w)^* \partial_\alpha \varphi - \omega^2 (B_e w)^* \varphi - f_a^* \varphi \right) dy \right| \\
&= 0.
\end{aligned}$$

Next, by (3.14) and using (3.36) and (3.37), together with integration by parts, we have

$$\begin{aligned}
(3.38) \quad & 0 = \int_{\mathbf{B}_2} \left(\sum_{\alpha, \beta \leq 1} (A_e^{\alpha\beta} \partial_\beta u)^* \partial_\alpha \varphi - \omega^2 (B_e \varphi)^* \varphi - f_e^* \phi \right) dy \\
&= \int_{\mathbf{B}_2 \setminus \mathbf{S}_1} \left(\sum_{\alpha, \beta \leq 1} (A_e^{\alpha\beta} \partial_\beta u)^* \partial_\alpha \varphi - \omega^2 (B_e \varphi)^* \varphi - f_e^* \phi \right) dy \\
&= \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbf{B}_2 \setminus (\Sigma_\varepsilon^+ \cup \mathbf{S}_1 \cup \Sigma_\varepsilon^-)} \left(\sum_{\alpha, \beta \leq 1} (A_e^{\alpha\beta} \partial_\beta u)^* \partial_\alpha \varphi - \omega^2 (B_e \varphi)^* \varphi - f_e^* \phi \right) dy \\
&= \lim_{\varepsilon \rightarrow 0^+} \left(\int_{\Upsilon_\varepsilon^+} - \left(\sum_{\alpha, \beta \leq 1} \nu_\alpha A_c^{\alpha\beta} \partial_\beta u \right)^* \varphi dS(y) + \int_{\Upsilon_\varepsilon^-} - \left(\sum_{\alpha, \beta \leq 1} \nu_\alpha A_a^{\alpha\beta} \partial_\beta u \right)^* \varphi dS(y) \right)
\end{aligned}$$

where, by a bit abuse of notations, ν denotes the exterior unit normal to respective domain, Σ_ε^+ and Σ_ε^- . Next, we estimate the integral in the right hand side of the last equation in (3.38). In the sequel, we denote by $\tilde{\nu}$ the exterior unit normal vector to the domain B_ε . Using the change of variables

in integration and the fact that $\tilde{v} = F^*(u|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1})$ belongs to $H^1(\mathbf{B}_2)^m$ and satisfies (3.16), we have

$$\begin{aligned}
 (3.39) \quad & \left| \int_{\Upsilon_\varepsilon^+} \left(\sum_{\alpha, \beta \leq 1} \nu_\alpha A_c^{\alpha\beta} \partial_\beta u \right)^* \varphi dS(y) \right| \\
 &= \left| \int_{\partial \mathbf{B}_\varepsilon} \left(\sum_{\alpha, \beta \leq 1} \tilde{\nu}_\alpha A_b^{\alpha\beta} \partial_\beta \tilde{v} \right)^* \tilde{\varphi} dS(x) \right| \\
 &= \left| \int_{\mathbf{B}_\varepsilon} \left(\sum_{\alpha, \beta \leq 1} (A_b^{\alpha\beta} \partial_\beta \tilde{v})^* \partial_\alpha \tilde{\varphi} - \omega^2 (B_b \tilde{v})^* \tilde{\varphi} - f_b^* \tilde{\varphi} \right) dx \right| \\
 &\lesssim \left(\|\tilde{v}\|_{H^1(\mathbf{B}_\varepsilon)^m} + \|f_b\|_{\tilde{H}^{-1}(\mathbf{B}_\varepsilon)^m} \right) \|\tilde{\varphi}\|_{H^1(\mathbf{B}_\varepsilon)^m} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+.
 \end{aligned}$$

By (3.38) and (3.39), we see

$$\int_{\mathbf{S}_1} \left(\sum_{\alpha, \beta \leq 1} \nu_\alpha A_a^{\alpha\beta} \partial_\beta w \right)^* \varphi dS(y) = \lim_{\varepsilon \rightarrow 0^+} \int_{\Upsilon_\varepsilon^-} \left(\sum_{\alpha, \beta \leq 1} \nu_\alpha A_a^{\alpha\beta} \partial_\beta w \right)^* \varphi dS(y) = 0$$

which implies by the fact $\varphi|_{\mathbf{S}_1}$ could be an arbitrary constant vector from \mathbb{C}^m that

$$\int_{\mathbf{S}_1} \sum_{\alpha, \beta \leq 1} \nu_\alpha A_a^{\alpha\beta} \partial_\beta w dS(y) = 0.$$

Now, let $\tilde{v} \in H^1(\mathbf{B}_2)^m$ and $w \in H^1(\mathbf{B}_1)^m$ be solutions, respectively to (3.16) and (3.17). Set u be $(F^{-1})^* \tilde{v}$ on $\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1$ and be w on \mathbf{B}_1 , and extend it to B_2 by setting it be zero on \mathbf{S}_1 . By Lemma 3.6, we see that $u \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$. Moreover, it is readily seen that one also has

$$(3.40) \quad \mathcal{P}_{[A_e, B_e]} u = f_c \quad \text{on } \mathbf{B}_2 \setminus \bar{\mathbf{B}}_1, \quad u|_{\mathbf{S}_2} = h,$$

and

$$(3.41) \quad \mathcal{P}_{[A_a, B_a]} u = f_a \quad \text{on } \mathbf{B}_1.$$

By Lemma 3.1, together with (3.40) and (3.41), we have for any $\varphi \in \mathcal{T}_0^\infty(\mathbf{B}_2)^m$

$$\begin{aligned}
 (3.42) \quad & \mathcal{Q}_{[A_e, B_e]}(u, \varphi) - \langle f_e, \varphi \rangle_\Omega \\
 &= \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbf{B}_2 \setminus (\Sigma_\varepsilon^+ \cup \mathbf{S}_1 \cup \Sigma_\varepsilon^-)} \left(\sum_{\alpha, \beta \leq 1} (A_e^{\alpha\beta} \partial_\beta u)^* \partial_\alpha \varphi - \omega^2 (B_e \varphi)^* \varphi - f_e^* \varphi \right) dy \\
 &= \lim_{\varepsilon \rightarrow 0^+} \left(\int_{\Upsilon_\varepsilon^+} - \left(\sum_{\alpha, \beta \leq 1} \nu_\alpha A_c^{\alpha\beta} \partial_\beta u \right)^* \varphi dS(y) + \int_{\Upsilon_\varepsilon^-} - \left(\sum_{\alpha, \beta \leq 1} \nu_\alpha A_a^{\alpha\beta} \partial_\beta u \right)^* \varphi dS(y) \right)
 \end{aligned}$$

Again, using the estimate in (3.39), we know

$$(3.43) \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\Gamma_\varepsilon^+} \left(\sum_{\alpha, \beta \leq 1} \nu_\alpha A_c^{\alpha\beta} \partial_\beta u \right)^* \varphi \, dS(y) = 0$$

On the other hand, by noting $u = w$ on \mathbf{B}_1 , and using (3.18) and $\varphi|_{\mathbf{S}_1}$ is a constant vector in \mathbb{C}^m , we further have

$$(3.44) \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\Gamma_\varepsilon^-} \left(\sum_{\alpha, \beta \leq 1} \nu_\alpha A_a^{\alpha\beta} \partial_\beta u \right)^* \varphi \, dS(y) = 0$$

Finally, by (3.42)–(3.44), we have

$$\mathcal{Q}_{[A_e, B_e]}(u, \varphi) = \langle f_e, \varphi \rangle_\Omega, \quad \forall \varphi \in \mathcal{T}_0^\infty(\mathbf{B}_2).$$

That is, $u \in H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$ is a solution to (3.14).

The proof is complete. \square

In the rest of this section, we study the interior problem (3.17)–(3.18). To that end, we introduce the following closed subspace of $H^1(\mathbf{B}_1)^m$,

$$(3.45) \quad \mathcal{W} := \{g \in H^1(\mathbf{B}_1)^m; \gamma g|_{\mathbf{S}_1} = \text{constant}\}.$$

Then, (3.17)–(3.18) is weakly formulated as

$$(3.46) \quad w \in \mathcal{W} \quad \text{and} \quad \mathcal{Q}_{[A_a, B_a]}(w, g) = \langle f_a, g \rangle_{\mathbf{B}_1}, \quad \forall g \in \mathcal{W}.$$

The corresponding homogeneous problem is

$$(3.47) \quad w \in \mathcal{W} \quad \text{and} \quad \mathcal{Q}_{[A_a, B_a]}(w, g) = 0, \quad \forall g \in \mathcal{W},$$

and by noting (2.3), its adjoint problem is

$$(3.48) \quad v \in \mathcal{W} \quad \text{and} \quad \mathcal{Q}_{[A, B^*]}(v, g) = 0, \quad \forall g \in \mathcal{W}.$$

Theorem 3.7. *Let W denote the set of solutions to (3.47). Then, either (i) $W = \{0\}$; or (ii) $\dim W = n$ for some finite $n \geq 1$. In the case (i), the problem (3.47) is uniquely solvable. Whereas in case (ii), the homogeneous adjoint problem (3.48) also has exactly n linearly independent solutions, say $v_1, v_2, \dots, v_n \in \mathcal{W}$, and the inhomogeneous problem (3.46) is solvable iff*

$$(3.49) \quad \langle v_p, f_a \rangle = 0 \quad \text{for } 1 \leq p \leq n.$$

Proof. Consider the operator $\mathcal{L} : \mathcal{W} \rightarrow \mathcal{W}^*$ determined by $\mathcal{Q}_{[A_a, B_a]}$ in the standard way as

$$(\mathcal{L}g_1)(g_2) = \mathcal{Q}_{[A_a, B_a]}(g_1, g_2).$$

Let $\mathcal{H} = L^2(\mathbf{B}_1)^m$ act as the pivot space. Clearly, the inclusion $\mathcal{W} \subseteq \mathcal{H}$ is compact. So, \mathcal{L} is Fredholm with index 0. Furthermore, each distribution $f_a \in \tilde{H}^{-1}(\mathbf{B}_1)^m$ gives rise to a unique functional $F_a \in \mathcal{W}^*$, defined by $F_a(g) = \langle f_a, g \rangle_{\mathbf{B}_1}$ for $g \in \mathcal{W}$. Thus, the equation (3.46) is equivalent to

$$(3.50) \quad \mathcal{L}w = F_a.$$

The celebrated Fredholm theory applied to (3.50) gives the desired results in the theorem. \square

Remark 3.8. In order to guarantee the existence of solutions to (3.1), we impose the following compatibility condition on the interior source/sink f_a ,

$$(3.51) \quad \langle v, f_a \rangle = 0,$$

where v is any solution to (3.48).

Concerning the solution to the cloaking problem (3.1), we have the following qualitative observations

Remark 3.9. Let $u \in H^1_{[A_e, B_e]}(\mathbf{B}_2)^m$ be a solution to (3.1). Set γ^+ and γ^- be the two sided trace operators on the cloaking \mathbf{S}_1 . Then, by (3.38), we see

$$\begin{aligned} & \int_{\mathbf{S}_1} \left[\sum_{\alpha, \beta \leq 1} \nu_\alpha A_e^{\alpha\beta} \partial_\beta u \right] dS \\ &= \int_{\mathbf{S}_1} [\gamma^+ \left(\sum_{\alpha, \beta \leq 1} \nu_\alpha A_e^{\alpha\beta} \partial_\beta u \right) - \gamma^- \left(\sum_{\alpha, \beta \leq 1} \nu_\alpha A_e^{\alpha\beta} \partial_\beta u \right)] dS \\ &= 0 \end{aligned}$$

On the other hand, we see that generically one has

$$[u]|_{\mathbf{S}_1} := \gamma^+ u - \gamma^- u = \text{constant vector} \neq 0,$$

since otherwise we would have an over-determined system (3.17)–(3.18). This observation also encompasses the necessity of introducing the finite energy solutions other than spatial H^1 -solutions to the singular cloaking problem (3.1).

4. GENERAL INVISIBILITY CLOAKING

Up till now, we have considered invisibility cloaking where the cloaked region is the unit disc \mathbf{B}_1 , and the cloaking layer is the annulus $\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1$. In this section, we extend our study to the general cloaking. To that end, let $G : \mathbf{B}_2 \rightarrow \Omega$ be an orientation-preserving and bi-Lipschitz mapping and let $D = G(\mathbf{B}_1)$. Then, for F in (2.15), let

$$\mathbf{K} = G \circ F \circ G^{-1} : \Omega \rightarrow \Omega,$$

which blows up the point $G(\{0\})$ to D within Ω while keeps $\partial\Omega$ fixed. Now, we shall show

Theorem 4.1. *Let $\{\Omega; \mathbf{A}_b, \mathbf{B}_b, \mathbf{f}_b\}$ with \mathbf{f}_b supported away from $G(\{0\})$ be the regular reference/background space. Then*

$$\{\Omega \setminus \bar{D}; \mathbf{A}_c, \mathbf{B}_c, \mathbf{f}_c\} = \mathbf{K}_* \{\Omega \setminus G(\{0\}); \mathbf{A}_b, \mathbf{B}_b, \mathbf{f}_b\}$$

is an invisibility cloaking device for the region D with respect to the reference space $\{\Omega; \mathbf{A}_b, \mathbf{B}_b, \mathbf{f}_b\}$. That is, for any extended object

$$\{\Omega; \mathbf{A}_e, \mathbf{B}_e, \mathbf{f}_e\} = \begin{cases} \{\Omega \setminus \bar{D}; \mathbf{A}_c, \mathbf{B}_c, \mathbf{f}_c\} & \text{in } \Omega \setminus \bar{D}, \\ \{D; \mathbf{A}_a, \mathbf{B}_a, \mathbf{f}_a\} & \text{in } D, \end{cases}$$

where $\{D; \mathbf{A}_a, \mathbf{B}_a\}$ is an arbitrary regular medium and $\mathbf{f}_a \in \tilde{H}^{-1}(D)^m$, we have

$$(4.1) \quad \Lambda_{\mathbf{A}_e, \mathbf{B}_e, \mathbf{f}_e}^\omega = \Lambda_{\mathbf{A}_b, \mathbf{B}_b, \mathbf{f}_b}^\omega.$$

Proof. Let

$$(4.2) \quad \{\mathbf{B}_2; A_b, B_b, f_b\} := (G^{-1})_* \{\Omega; \mathbf{A}_b, \mathbf{B}_b, \mathbf{f}_b\},$$

and

$$(4.3) \quad \begin{aligned} \{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1; A_c, B_c, f_c\} &= (F \circ G^{-1})_* \{\Omega \setminus G(\{0\}); \mathbf{A}_b, \mathbf{B}_b, \mathbf{f}_b\} \\ &= F_* \{\mathbf{B}_2 \setminus \{0\}; A_b, B_b, f_b\}. \end{aligned}$$

Then,

$$\{\Omega \setminus \bar{D}; \mathbf{A}_c, \mathbf{B}_c, \mathbf{f}_c\} = G_* \{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1; A_c, B_c, f_c\}.$$

Since G is orientation-preserving and bi-Lipschitz, we know $\{\mathbf{B}_2; A_b, B_b, f_b\}$ is a regular space and, $\mathbf{A}_c, \mathbf{B}_c$ have same singularities on the cloaking interface ∂D^+ as of A_c, B_c on \mathbf{S}_1^+ . In order to stick close to our discussion in Sections 2 and 3, we introduce

$$\{\mathbf{B}_1; A_a, B_a, f_a\} = (G^{-1})_* \{D; \mathbf{A}_a, \mathbf{B}_a, \mathbf{f}_a\},$$

and define $\{\mathbf{B}_2; A_e, B_e, f_e\}$ correspondingly. Now we introduce the singularly weighted Sobolev space $H_{[\mathbf{A}_e, \mathbf{B}_e, \mathbf{f}_e]}^1(\Omega)^m$ similarly to $H_{[A_e, B_e]}^1(\mathbf{B}_2)^m$. The solution to the cloaking problem,

$$(4.4) \quad \mathcal{P}_{[\mathbf{A}_e, \mathbf{B}_e]} \mathbf{u} = \mathbf{f}_e \quad \text{on } \Omega, \quad \mathbf{u}|_{\partial\Omega} = \mathbf{h}$$

can be defined in a similar manner as in (3.14). It is readily seen that $\mathbf{u} \in H_{[\mathbf{A}_e, \mathbf{B}_e]}^1(\Omega)^m$ is a solution to (4.4) iff $u := G^* \mathbf{u} \in H_{[A_e, B_e]}^1(\mathbf{B}_2)$ is a solution to

$$(4.5) \quad \mathcal{P}_{[A_e, B_e]} u = f_e \quad \text{on } \mathbf{B}_2, \quad u|_{\mathbf{S}_2} = h$$

where $h = (G|_{\mathbf{S}_2})^* \mathbf{h}$. By Theorem 3.4, we know $\tilde{v}(x) = (F^*(u|_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1}))^e \in H^1(\mathbf{B}_2)^m$ satisfies

$$(4.6) \quad \mathcal{P}_{[A_b, B_b]} \tilde{v} = f_b \quad \text{on } \mathbf{B}_2, \quad \tilde{v}|_{\mathbf{S}_2} = h,$$

whereas $w = u|_{\mathbf{B}_1} \in H^1(\mathbf{B}_1)^m$ is a solution to

$$(4.7) \quad \mathcal{P}_{[A_a, B_a]} w = f_a \quad \text{on } \mathbf{B}_1, \quad w|_{\mathbf{S}_1} = c_0,$$

with $c_0 \in \mathbb{C}^m$ a constant vector determined by

$$(4.8) \quad \int_{\mathbf{S}_1} \sum_{\alpha, \beta \leq 1} \nu_\alpha A_a^{\alpha\beta} \partial_\beta w \, dS = 0.$$

Therefore, we know $\tilde{\mathbf{v}} := (G^{-1})^* \tilde{v} = (\mathbf{K}^*(\mathbf{u}|_{\Omega \setminus \bar{D}}))^e \in H^1(\Omega)^m$ satisfies

$$(4.9) \quad \mathcal{P}_{[\mathbf{A}_b, \mathbf{B}_b]} \tilde{\mathbf{v}} = \mathbf{f}_b \quad \text{on } \Omega, \quad \tilde{\mathbf{v}}|_{\partial\Omega} = \mathbf{h},$$

whereas $\mathbf{w} := (G^{-1})^* w = \mathbf{u}|_D \in H^1(D)^m$ is a solution to

$$\mathcal{P}_{[\mathbf{A}_a, \mathbf{B}_a]} \mathbf{w} = \mathbf{f}_a \quad \text{on } D, \quad \mathbf{w}|_D = \mathbf{c}_0,$$

with $\mathbf{c}_0 \in \mathbb{C}^m$ a constant vector determined by

$$\int_{\partial D} \sum_{\alpha, \beta \leq 1} \nu_\alpha \mathbf{A}_\alpha^{\alpha\beta} \partial_\beta \mathbf{w} \, dS = 0.$$

Finally, by a similar argument to that for the proof of Theorem 2.3, one can show (4.1). The proof is completed. \square

5. DISCUSSION

In this paper, we investigate the invisibility cloaking for a general system of PDEs in \mathbb{R}^2 . The material parameters possess transformation properties, which allow the construction of cloaking media by singular push-forward transformations. The cloaking media are inevitably singular resulting in singular PDEs. Thus one needs to be careful in defining the meaning of solutions to the singular cloaking problems. In [3] the notion of finite energy solutions was introduced which is a meaningful physical condition because it relates to the energy of the waves. We follow their approach to treat more singular PDEs in the present paper. By studying the finite energy solution, we decouple the cloaking problem with one in the cloaking region having the same DtN operator as that in the background/reference space, and the other one in the cloaked region possessing some ‘hidden’ boundary condition on the interior cloaking interface. On the other hand, we would like to point out that this is not the unique way of defining solutions to the singular cloaking problems. An alternative treatment is provided in [7] by the authors to split the finite energy solutions space directly and hence decouple the cloaking problem into the cloaked region and cloaking region automatically. In order to achieve more physical insights into these solutions, one might relate them to solutions from regularized approximate cloaking.

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